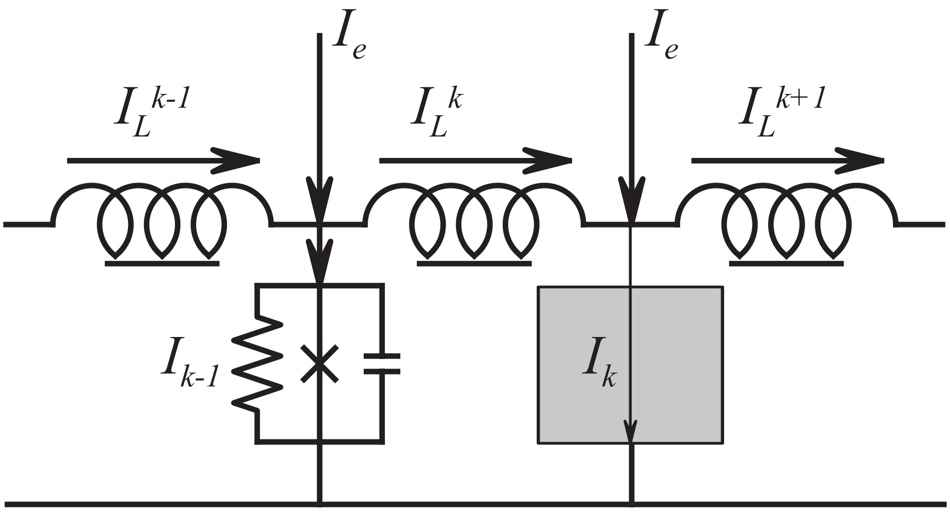
Lecture 1: sine-Gordon equation and solutions

- Equivalent circuit
- Derivation of sine-Gordon equation
- The most important solutions
  - ☐ plasma waves
  - ☐ a soliton!
  - ☐ chain of solitons
  - ☐ resistive state
  - ☐ breather and friends
- Mechanical analog: the chain of pendula
- Penetration of magnetic field

Why LJJ?

- Almost ideal system to study soliton dynamics (simple measurable quantities e.g. $V \propto u$)
- Applications as oscillators (FF, ZFS, FS, Cherenkov, FF transistors)
- Physics of layered HTS (dynamics+losses)
- Studying “fine” properties: fluxon in a potential, energy level quantization, etc.
- Some JJ are just long
- It is nice non-linear physical system ;-)

Equation of long Josephson junction



$$\begin{cases} \phi_{k+1} - \phi_k & = 2\pi\Phi_k/\Phi_0 = \frac{2\pi}{\Phi_0}\left(\Phi_e - LI_L^k\right) \\ I_L^k + I_e & = I_L^{k+1} + I_k \end{cases}$$

$$I_{k,e} \to J_{k,e}dx, \qquad L \to ldx, \qquad \Phi_e \to H\Lambda dx$$

$$\begin{cases} \phi_x = \frac{2\pi}{\Phi_0}\left(H\Lambda - lI_L^k\right) \\ \frac{I_L^k}{dx} = J_e - J_k \end{cases} \quad \to \quad \begin{matrix} @\ x = 0, L\ I_L^k = 0 \\ \phi_x = \frac{2\pi H\Lambda}{\Phi_0} \end{matrix}$$

We get rid of I_L^k: $\boxed{\frac{\Phi_0}{2\pi l}\phi_{xx} = J(x) - J_e}$

Sine-Gordon Equation

RSJ model & $l = \mu_0 d'/w$, $J(x) = j(x)w$, $J_e = j_e w$:

$$\frac{\Phi_0}{2\pi\mu_0 d'}\phi_{xx} = j_c \sin\phi + \frac{V}{R} + CV_t - j_e$$

$$\underbrace{\frac{\Phi_0}{2\pi\mu_0 d' j_c}}_{\lambda_J^2}\phi_{xx} = \sin\phi + \underbrace{\frac{\Phi_0}{2\pi R j_c}}_{\omega_c^{-1}}\phi_t + \underbrace{\frac{\Phi_0 C}{2\pi j_c}}_{\omega_p^{-2}}\phi_{tt} - \underbrace{\frac{j_e}{j_c}}_{\gamma}$$

Normalized units: $\tilde{x} = x/\lambda_J$, $\tilde{t} = t\omega_p$.

$$\boxed{\phi_{\tilde{x}\tilde{x}} - \phi_{\tilde{t}\tilde{t}} - \sin\phi = \alpha\phi_{\tilde{t}} - \gamma}$$

perturbed sine-Gordon equation

$$\alpha = \frac{\omega_p}{\omega_c} = \frac{1}{\sqrt{\beta_c}} = \frac{1}{\sqrt{\dfrac{2\pi}{\Phi_0} j_c R^2 C}}$$

Other normalized quantities:

$$v = \phi_{\tilde{t}}, \quad h = \phi_{\tilde{x}}$$

Characteristic velocity: $\bar{c}_0 = \lambda_J \omega_p$, $\quad u = v/\bar{c}_0$

Boundary conditions

Boundary conditions for linear LJJ:

$$\phi_{\tilde{x}}|_{x=0,\ell} = h$$

Boundary conditions for annular LJJ:

$$\phi|_{\tilde{x}=0} = \phi|_{\tilde{x}=\ell} + 2\pi N$$

$$\phi_{\tilde{x}}|_{\tilde{x}=0} = \phi_{\tilde{x}}|_{\tilde{x}=\ell}$$

Perturbations are small: $\alpha \ll 1$, $\gamma \ll 1$.
For Nb-Al-AlO$_x$-Nb junctions at $T = 4.2\,\mathrm{K}$ $\alpha \sim 10^{-2}$.
The typical value of $\gamma \sim 0.1$.
Taking $\alpha = \gamma = 0$ we get:

$$\boxed{\phi_{\tilde{x}\tilde{x}} - \phi_{\tilde{t}\tilde{t}} - \sin\phi = 0}$$

unperturbed sine-Gordon equation

1: Josephson plasma waves

Consider small amplitude waves:

$$\phi(x,t) = A\sin(kx - \omega t), \quad A \ll 1$$

Substituting into $\phi_{xx} - \phi_{tt} - \sin\phi = 0$ and using approximation

$$\sin\left[A\sin(kx - \omega t)\right] \approx A\sin(kx - \omega t)$$

we get the dispersion relation for EM waves in the LJJ:

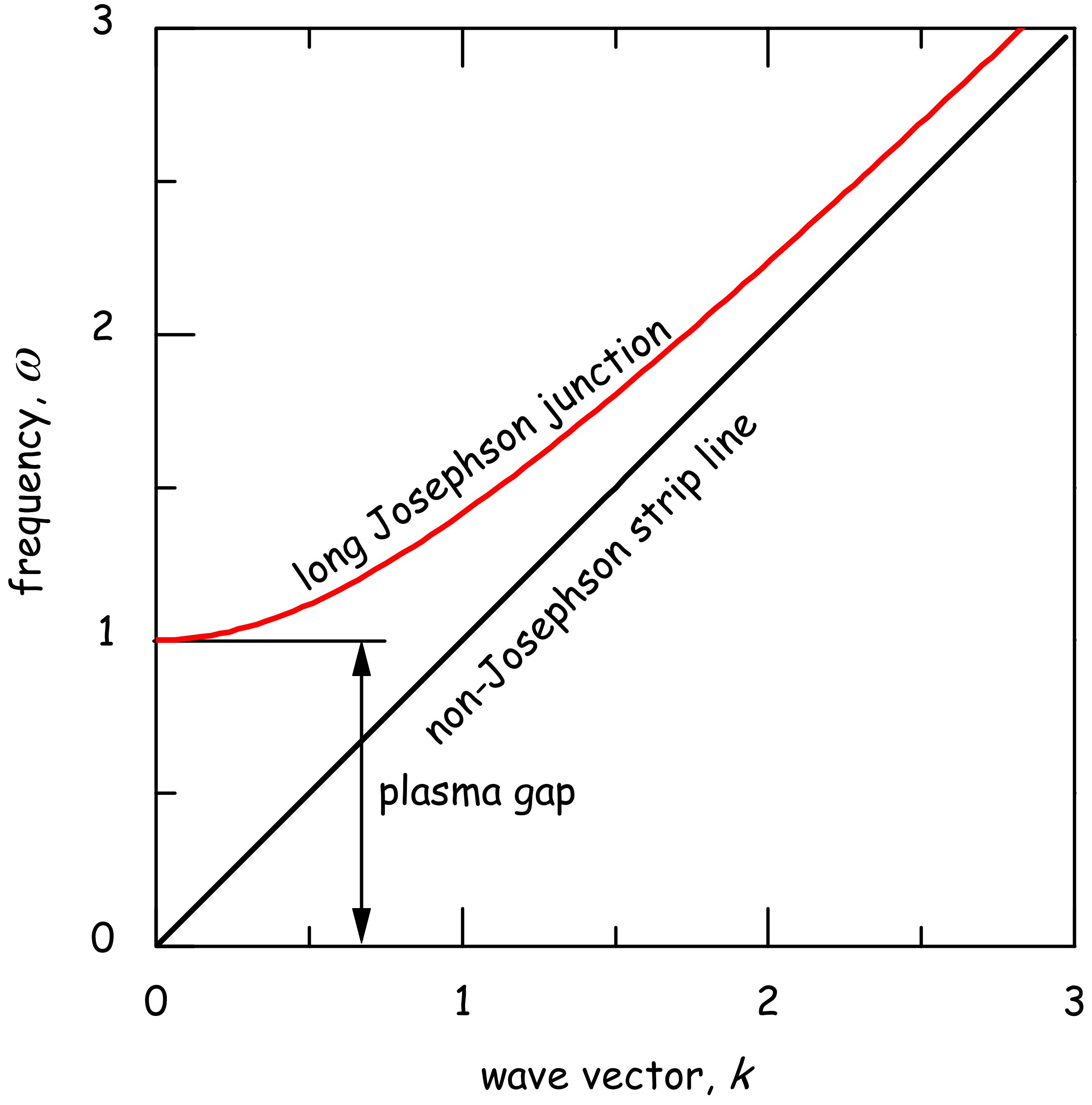
$$\boxed{\omega(k) = \sqrt{1 + k^2}}$$

Picture. Non-Josephson strip-line. $\bar{c}_0 = 1$ is the *Swihart velocity*. Plasma gap.
Phase velocity:

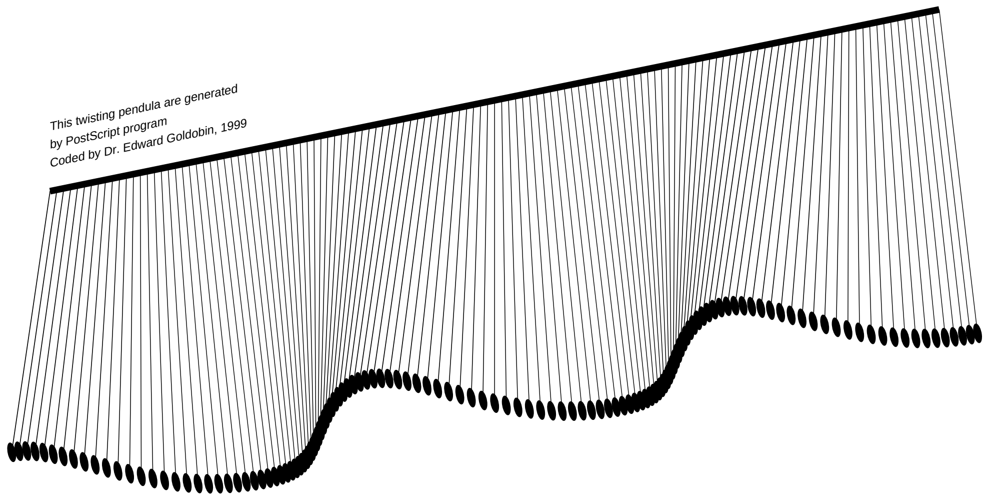
$$u_{ph} = \frac{\omega}{k} = \sqrt{1 + \frac{1}{k^2}} > 1$$

i.e. $u_{ph} > \bar{c}_0$ $\rightarrow$ Swihart velocity in the LJJ is the minimum phase velocity and maximum group velocity of linear EM waves.

Dispersion of linear waves



Mechanical analog of LJJ



Josephson phase	ϕ	angle of pendulum
bias current	γ	torque
damping coefficient	α	friction in the axis
Josephson voltage	ϕ_t	angular frequency

2: Soliton

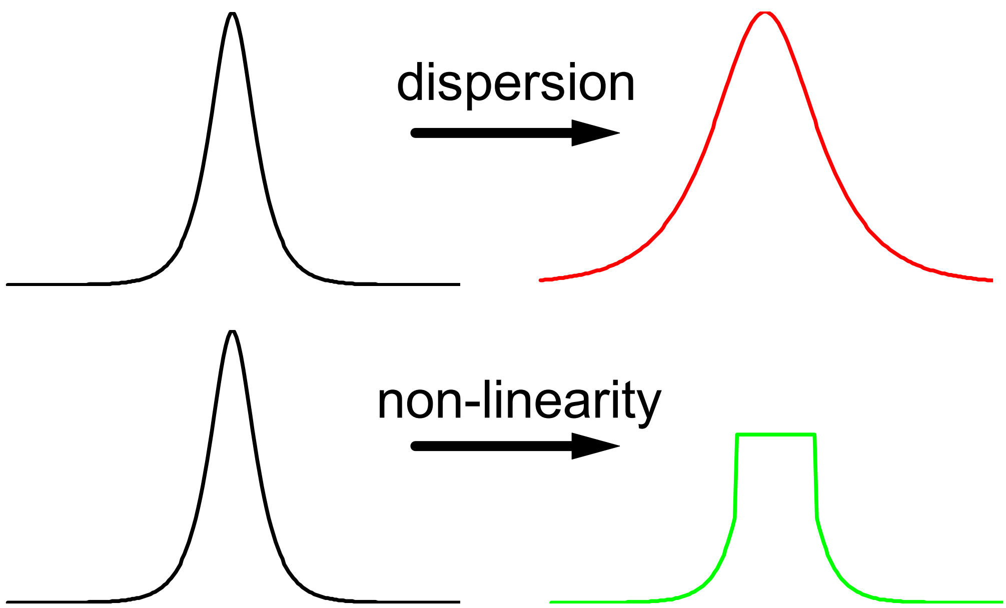
Unperturbed sine-Gordon equation has exact solution:

$$\phi(x,t) = 4\arctan\exp\left(\pm\frac{x-ut}{\sqrt{1-u^2}}\right)$$

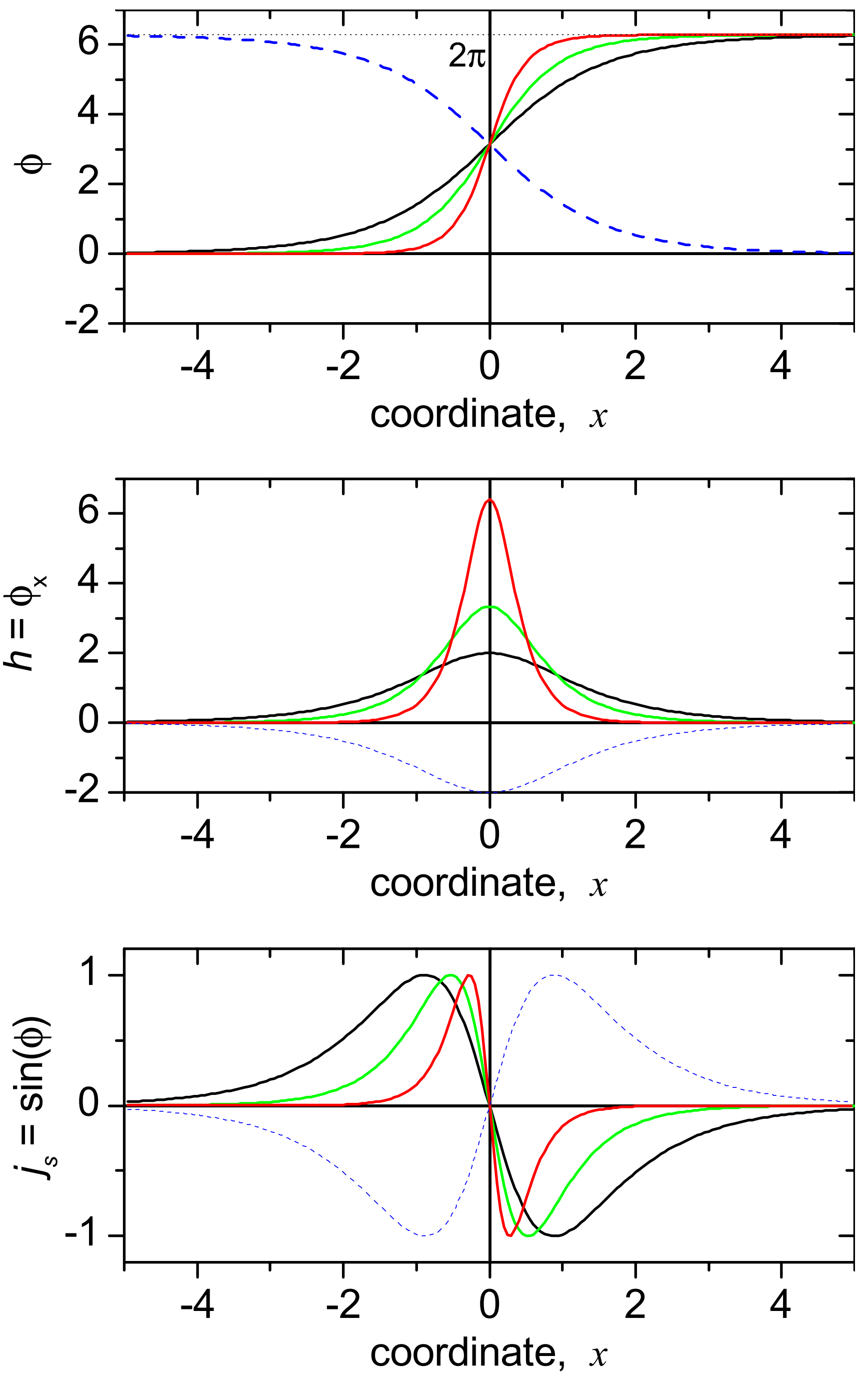
This is a solitary wave or *soliton*. It can move with velocity $0 \leq u < 1$ (i.e. $\bar{c}_0$!). Picture. Soliton is a kink which changes the Josephson phase from 0 to 2π (soliton) or from 2π to 0 (anti-soliton).
The field of soliton is

$$h = \phi_x = \frac{2}{\cosh(\frac{x-ut}{\sqrt{1-u^2}})}, \quad h|_{x=0} = 2$$

Existence of the soliton



Fluxon shape & contraction



Lorentz invariance

Sine-Gordon equation is invariant with respect to the Lorentz transformation:

$$x \to x' = \frac{x - ut}{\sqrt{1-u^2}}, \quad t \to t' = \frac{t - x/u}{\sqrt{1-u^2}}$$

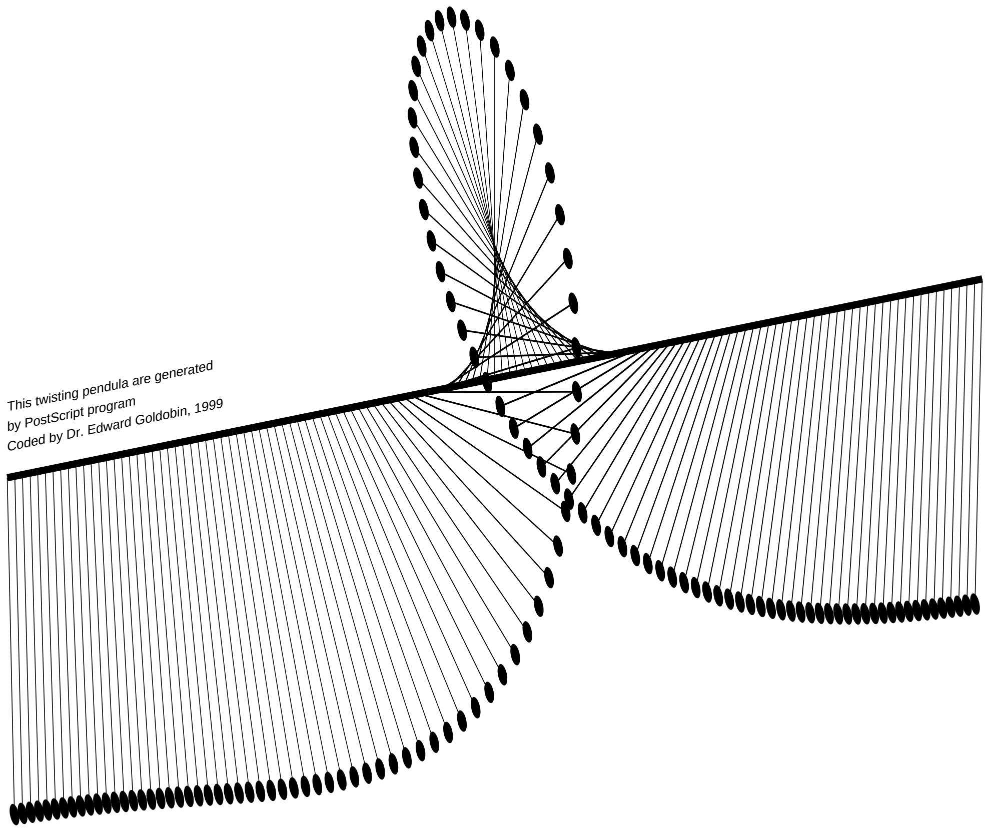
Thus, soliton behaves as relativistic object and contracts when approaching the velocity of (our!) light — Swihart velocity! Picture.
In spite of contraction, soliton always carries one quantum of magnetic flux:

$$\int_{-\infty}^{\infty} \phi_x \, dx = \phi(\infty) - \phi(-\infty) = 2\pi$$

Since $\phi = \frac{2\pi\Phi}{\Phi_0}$, $\Phi = \Phi_0$. Therefore, the soliton in LJJ is called *fluxon*. An antifluxon carries $-\Phi_0$.
The energy (mass) of the soliton (next slide):

$$E(u) = m(u) = \frac{8}{\sqrt{1-u^2}}$$

Mechanical analog of LJJ



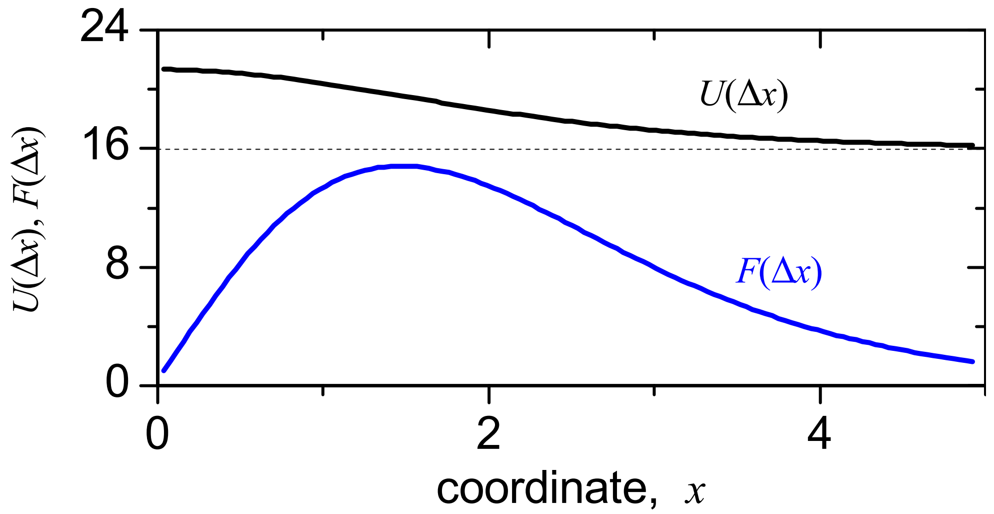
Josephson phase	ϕ	angle of pendulum
bias current	γ	torque
damping coefficient	α	friction in the axis
Josephson voltage	ϕ_t	angular frequency

Fluxon interaction

Hamiltonian (energy) of the LJJ:

$$H = \int_{-\infty}^{+\infty} \underbrace{\frac{\phi_t^2}{2}}_{K} + \underbrace{\frac{\phi_x^2}{2} + (1 - \cos\phi)}_{U} \, dx$$

Substituting two solitons with the distance Δx between them



- two fluxons *repel* each other.
- fluxon and anti-fluxon attract.

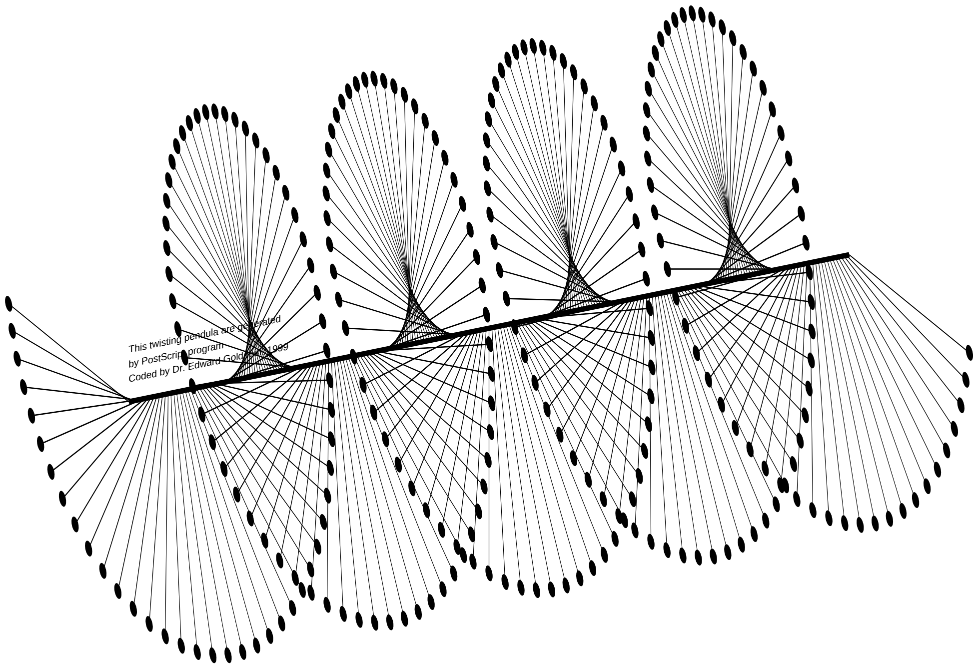
3: Chain of fluxons

Fluxons can form a dense chain

$$\phi(x,t) = 2\,\mathrm{am}(x - ut, k) + \pi$$

For $h \gg 1$:

$$\phi(x,t) \approx h(x - ut) - \frac{\sin\left[h(x-ut)\right]}{h^2\left(1-u^2\right)}$$

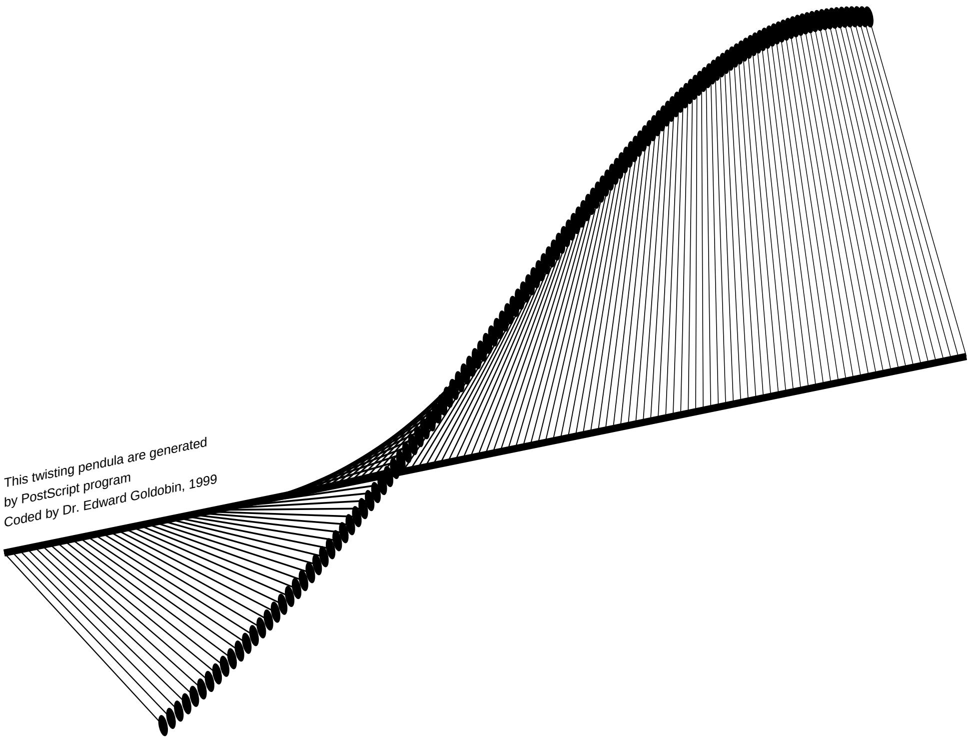


(-: mincing machine :-)

Intuitive explanation of repelling.

4: Resistive (McCumber) state

$$\phi(x,t) = (Hx + \omega t) - \frac{\sin(Hx + \omega t)}{\omega^2 - H^2}$$



5. Breather

Since fluxon and antifluxon attract each other, they can form a bound state which oscillates around common center of mass:

$$\phi(x,t) = 4\arctan\left[\tan\Theta \frac{\sin(t\cos\Theta)}{\cosh(x\sin\Theta)}\right]$$

where $\Theta = 0 \ldots \pi/2$. A breather with the moving center of mass can be obtained using Lorentz transformations.

Fluxon-antifluxon collision:

$$\phi(x,t) = 4\arctan\left[\frac{\sinh\left(\frac{ut}{\sqrt{1-u^2}}\right)}{u\cosh\left(\frac{x}{\sqrt{1-u^2}}\right)}\right]$$

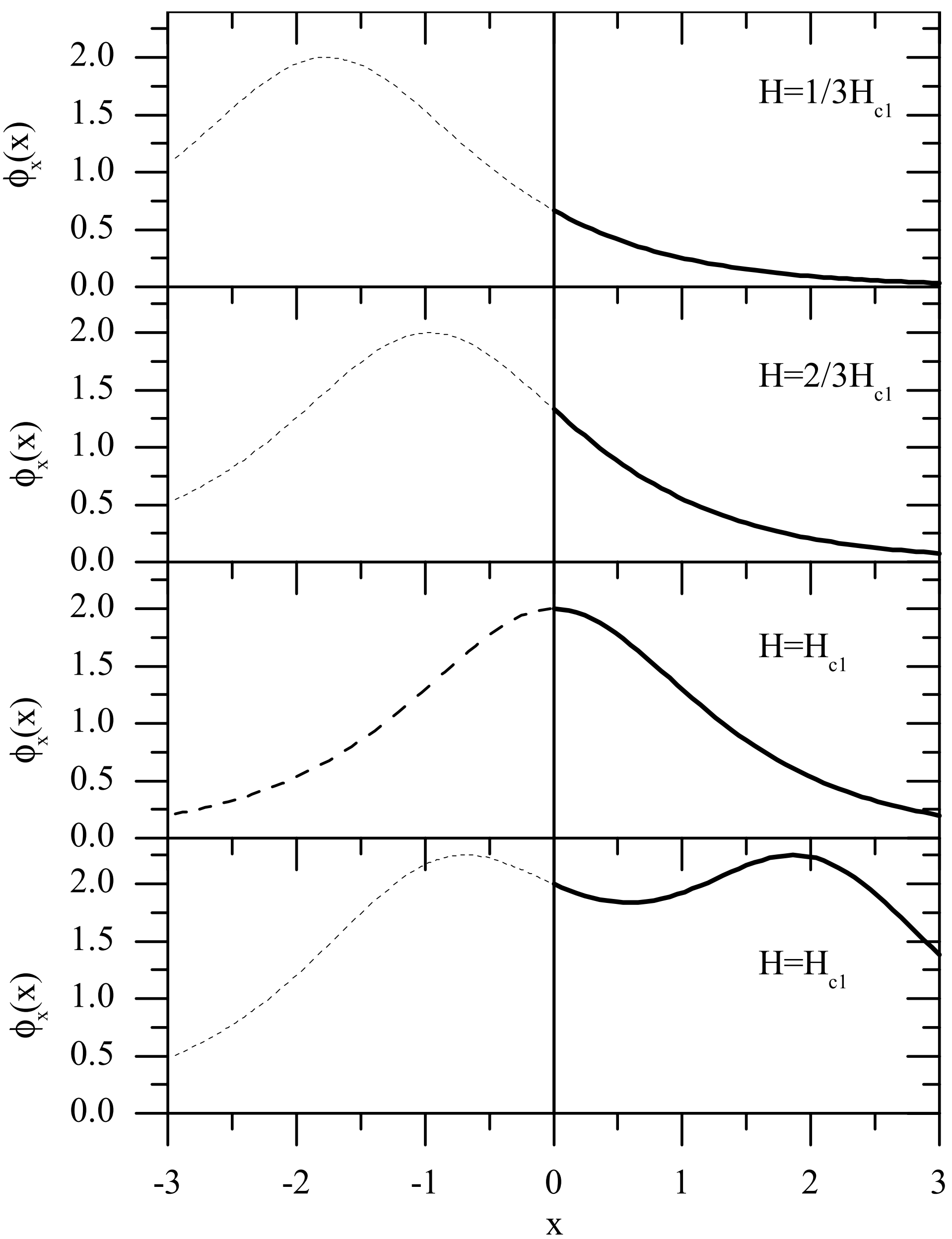
There is a positive phase-shift !

Fluxon-fluxon collision:

$$\phi(x,t) = 4\arctan\left[\frac{\sinh\left(\frac{x}{\sqrt{1-u^2}}\right)}{u\cosh\left(\frac{ut}{\sqrt{1-u^2}}\right)}\right]$$

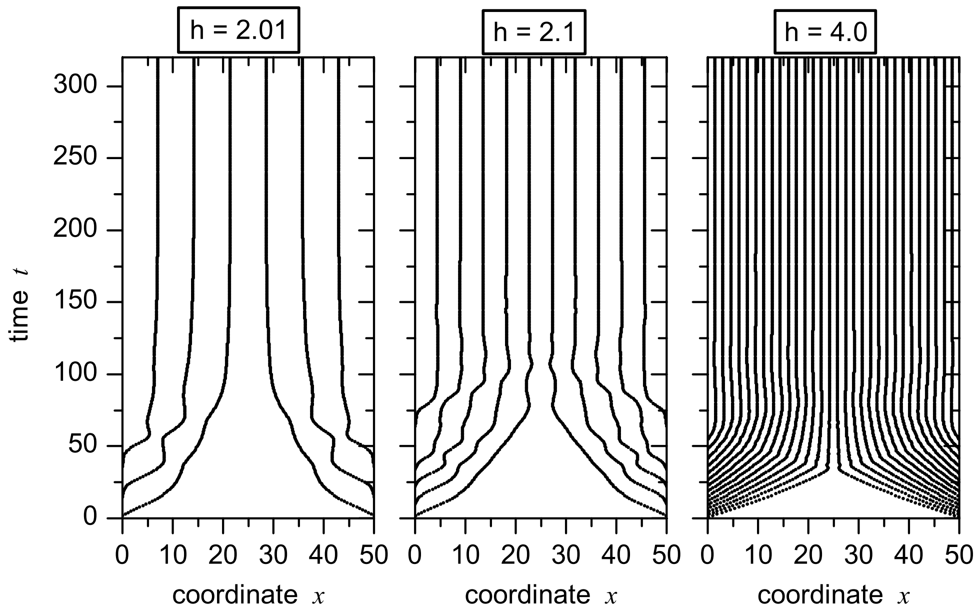
There is a negative phase-shift !

Penetration of magnetic field into LJJ



Penetration of magnetic field into LJJ

When h exceeds 2, the fluxons enter the junction and fill it with some density, forming a dense fluxon chain.



Example: $h = 4$, $\phi = hx$, so $\phi(L) - \phi(0) = h\ell$, $N = \frac{h\ell}{2\pi} = \frac{4\times 50}{6.28} \approx 31.8$. Looking at picture, we see 30 fluxons. For smaller fields the correspondence is worse, since the dense fluxon chain approximation works not so good, and at $h < 2$ does not work at all. e.g. for $h = 2.1$, $N = 16.7$, but we see only 10 fluxons.

Lecture 2: Dynamics of fluxon

- Perturbation theory
- Fluxon steps in annular LJJ
- ZFS in linear LJJ
- Flux-Flow and FFS (Eck peak)
- Fiske Steps

Perturbation theory of McLoughlin and Scott

All solutions of s-G equation (except resistive state) considered during the previous lection are solutions of the unperturbed sG equation:

$$\boxed{\phi_{xx} - \phi_{tt} - \sin\phi = 0}$$

We also have seen that:

$$\boxed{H = \int_{-\infty}^{+\infty} \underbrace{\frac{\phi_t^2}{2}}_{K} + \underbrace{\frac{\phi_x^2}{2} + (1 - \cos\phi)}_{U} \; dx} \qquad (1)$$

The real equation which governs the Josephson phase dynamics in the system is *perturbed* s-G equation:

$$\boxed{\phi_{xx} - \phi_{tt} - \sin\phi = \alpha\phi_t - \gamma} \qquad (2)$$

The Hamiltonian (1) corresponds only to the l.h.s. of (2) while r.h.s. describes the energy dissipation and injection.
Let us write down the change of energy with time.

Energy balance equations

$$\frac{dH}{dt} = \int_{-\infty}^{+\infty} \frac{d}{dt}\left[\frac{\phi_t^2}{2} + \frac{\phi_x^2}{2} + (1-\cos\phi)\right] dx$$

$$= \int_{-\infty}^{+\infty} (\phi_t\phi_{tt} + \phi_x\phi_{xt} + \phi_t \sin\phi)\ dx$$

$$= \underbrace{\phi_x\phi_t\big|_{-\infty}^{+\infty}}_{\text{zero if localized}} + \int_{-\infty}^{+\infty} (\phi_t\phi_{tt} - \phi_{xx}\phi_t + \phi_t \sin\phi)\ dx$$

$$= \int_{-\infty}^{+\infty} -\phi_t \underbrace{(\phi_{xx} - \phi_{tt} - \sin\phi)}_{\text{l.h.s. of sine-Gordon}}\ dx$$

$$= \int_{-\infty}^{+\infty} -\phi_t \underbrace{(\alpha\phi_t - \gamma)}_{\text{r.h.s. of sine-Gordon}}\ dx$$

$$= \boxed{\int_{-\infty}^{+\infty} \left(\gamma\phi_t - \alpha\phi_t^2\right)\ dx = 0}$$

Energy balance for fluxon

$$\underbrace{\int_{-\infty}^{+\infty} \gamma\phi_t \, dx}_{F_\gamma u} = \underbrace{\int_{-\infty}^{+\infty} \alpha\phi_t^2 \, dx}_{F_\alpha u}$$

$$\phi(x,t) = 4 \arctan \exp \frac{x - ut}{\sqrt{1-u^2}}$$

$$\phi_t(x,t) = \frac{-u}{\sqrt{1-u^2}} \frac{2}{\cosh \frac{x-ut}{\sqrt{1-u^2}}}$$

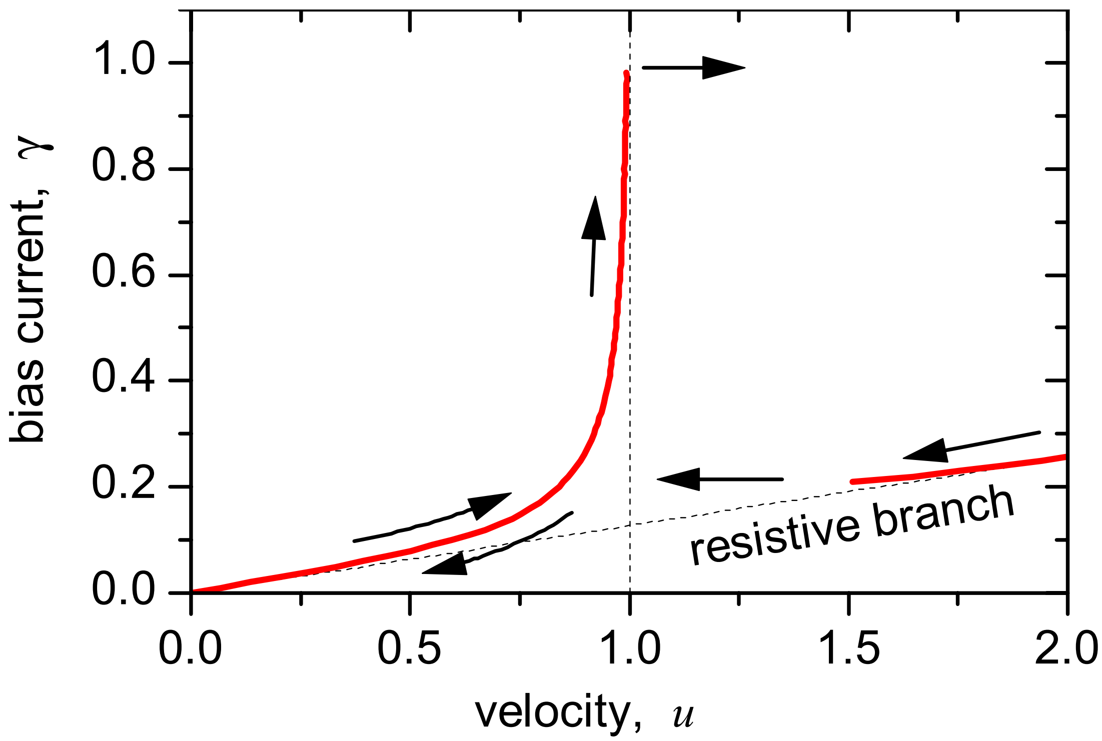
$$-\gamma 2\pi u = \alpha \frac{8u^2}{\sqrt{1-u^2}}$$

$$|u| = \frac{1}{\sqrt{1 + \left(\frac{4\alpha}{\pi\gamma}\right)^2}}$$

$$|u| \approx \frac{\pi\gamma}{4\alpha}, \quad \text{for } \gamma \ll 1$$

$$|u| \to 1, \quad \text{for } \gamma \to 1$$

I–V Characteristic



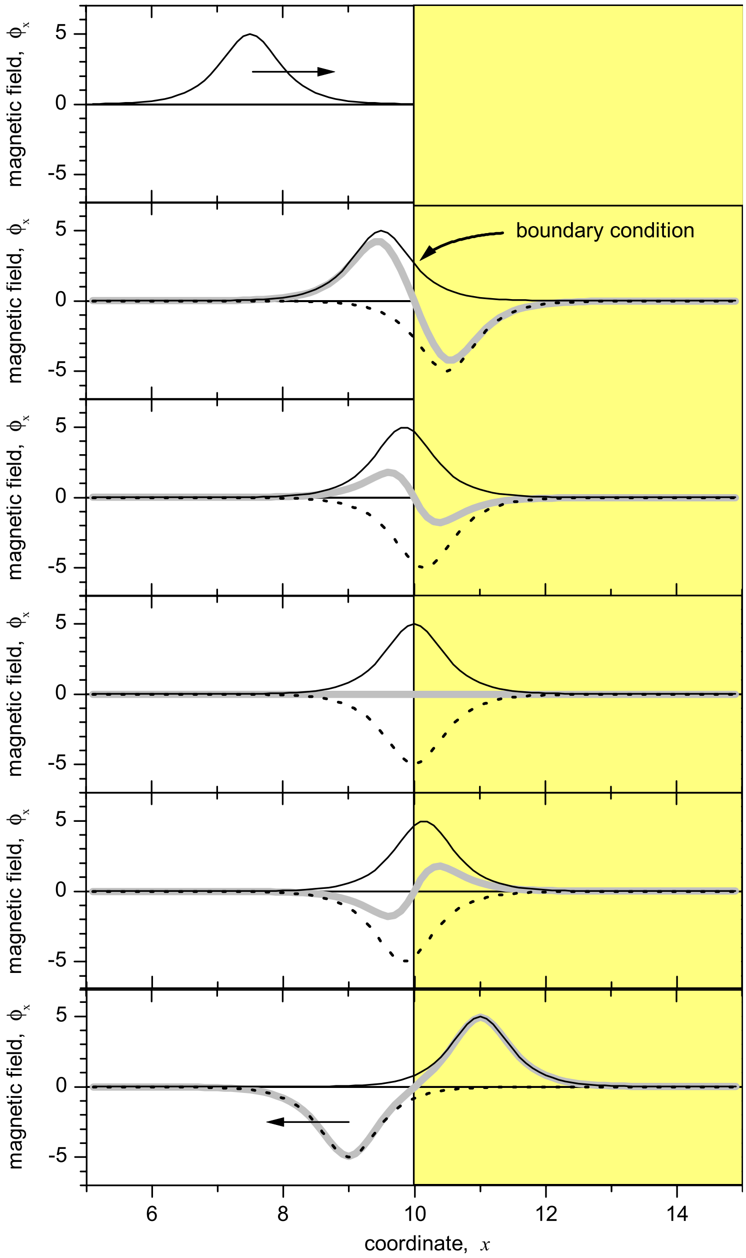
Example: annular LJJ

$$V = \frac{\Delta\Phi}{\Delta t} = \frac{n\Phi_0}{L/u} = \frac{n\Phi_0 u}{L}$$

I-V characteristic $\Longleftrightarrow$ γ–u characteristic

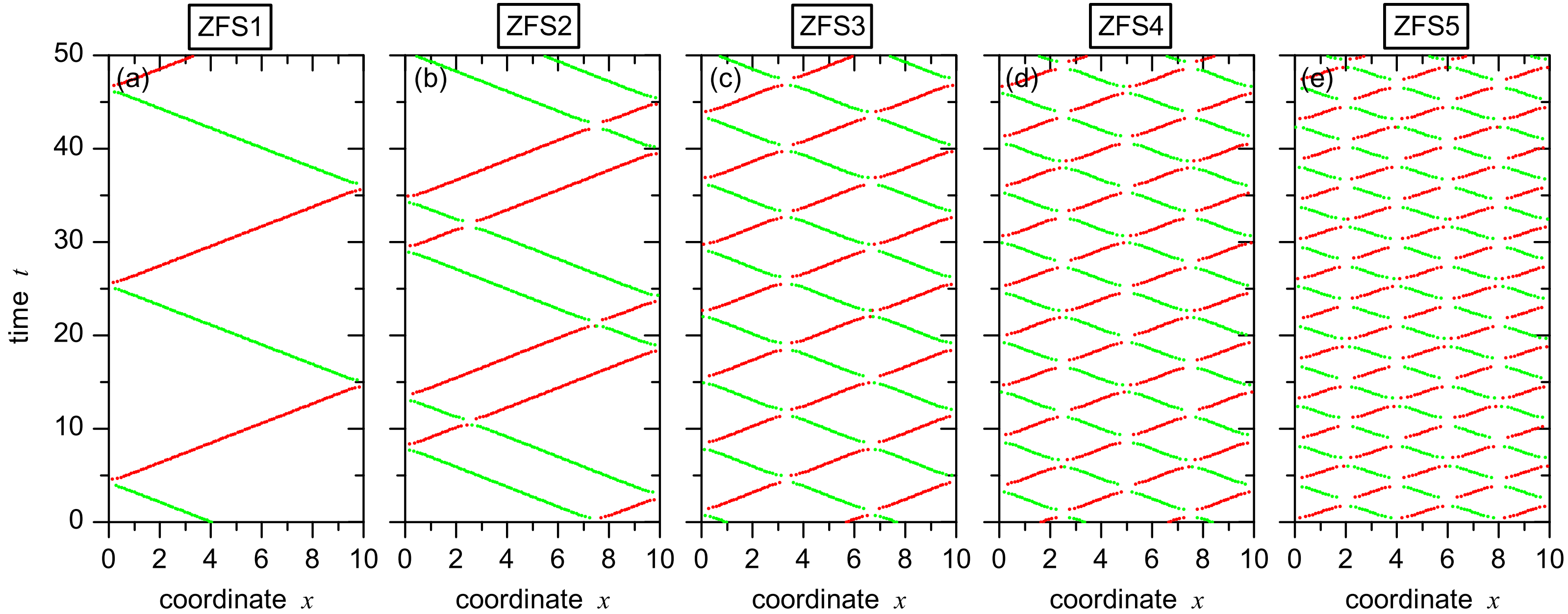
$$V_{\max} = \frac{n\Phi_0 \bar{c}_0}{L}$$

Collision with the edge



Collision with edge $\equiv$ fluxon-antifluxon collision

Fluxon trajectories

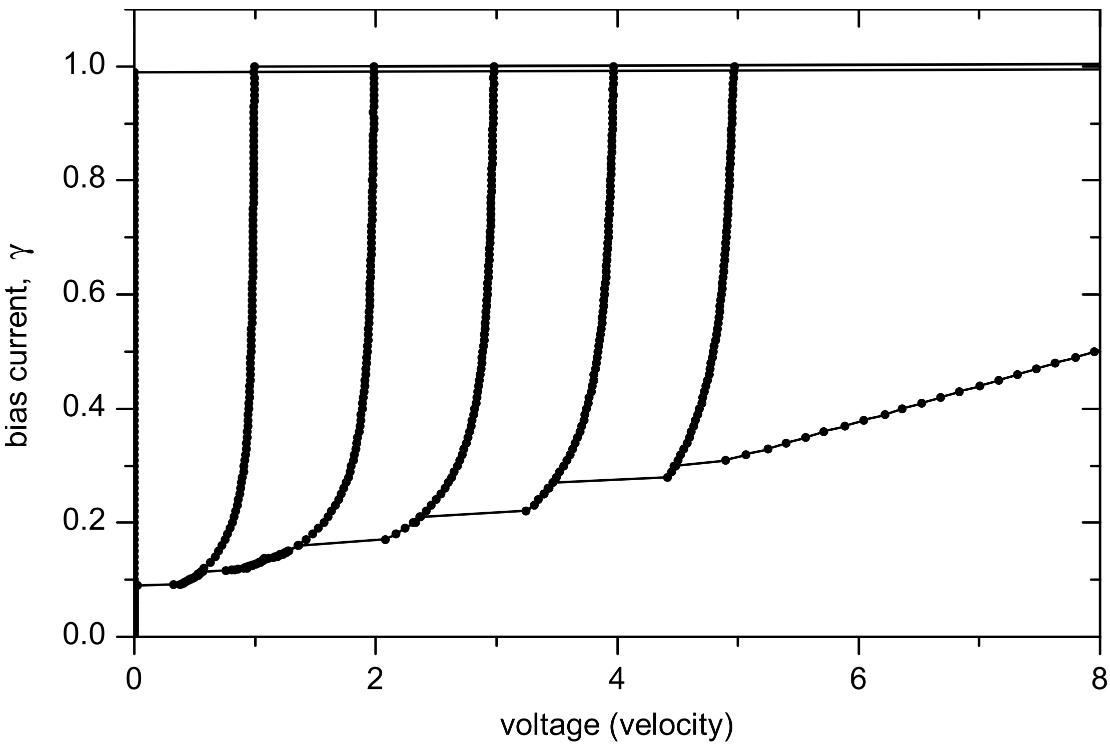


animations see at http://christo.pit.physik.uni-tuebingen.de:88/FluxonDynamics/

Zero Field Steps

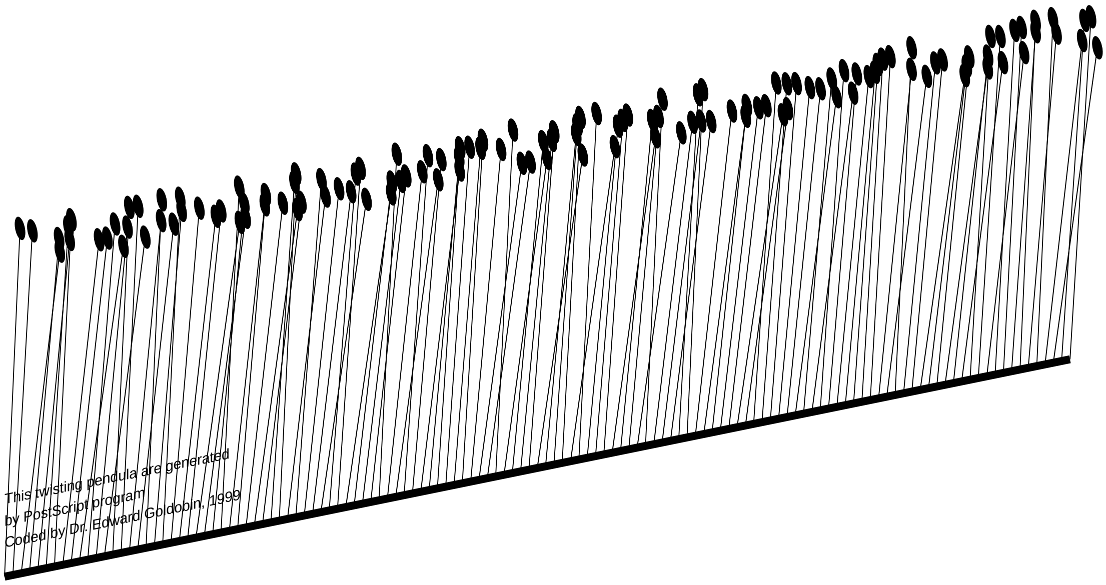
$$V = \frac{\Delta\Phi}{\Delta t} = \frac{\Phi_0 - (-\Phi_0)}{2L/u} = \frac{\Phi_0 u}{L}$$

But frequency of collisions is $f = u/2L$, $i.e.$, two times lower!



How the fluxons get into the junction?

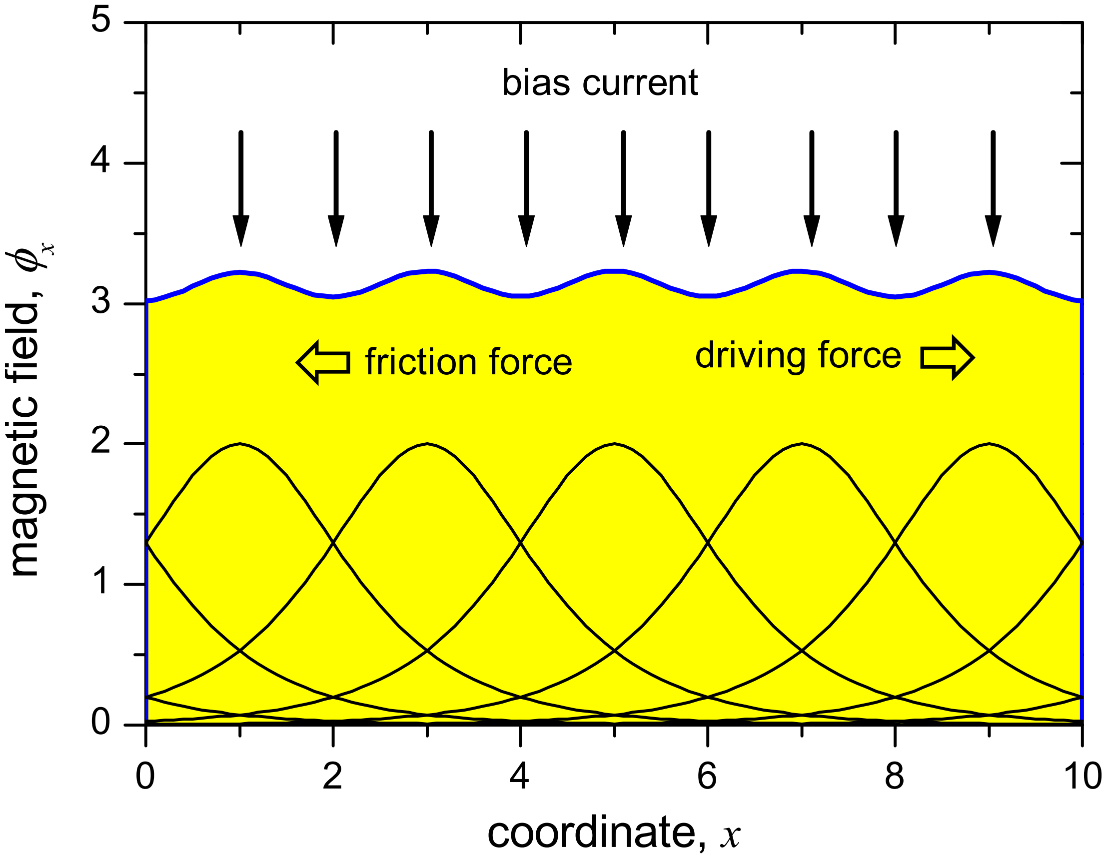
Moving down along McCumber branch the rotation frequency becomes lower resulting in instability due to thermal fluctuations.



ZFS are better visible at $T > 4.2\,\mathrm{K}$

Flux-flow

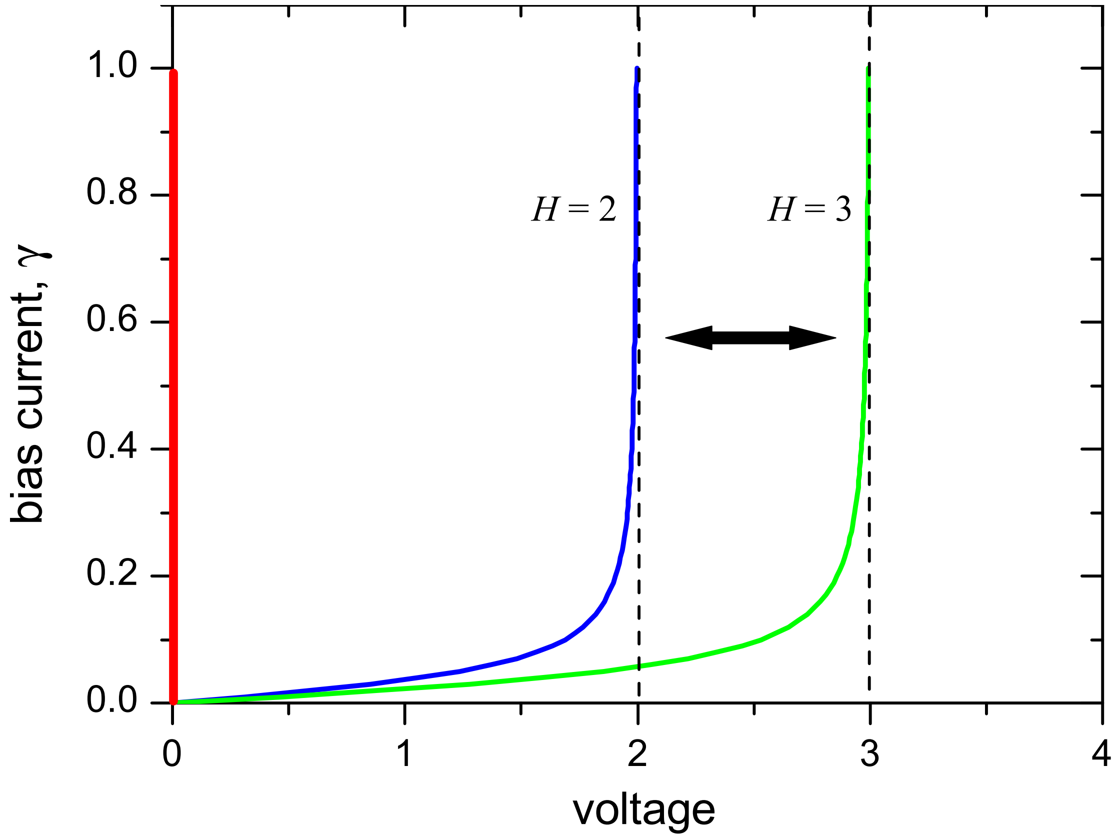
Let us suppose that LJJ is filled with fluxons e.g. some field $H > H_{c1}$ is applied to the linear LJJ.



$$V_{\mathrm{FF}} = \frac{\Delta\Phi}{\Delta t} = \frac{H\Lambda L}{L/u} = H\Lambda u$$

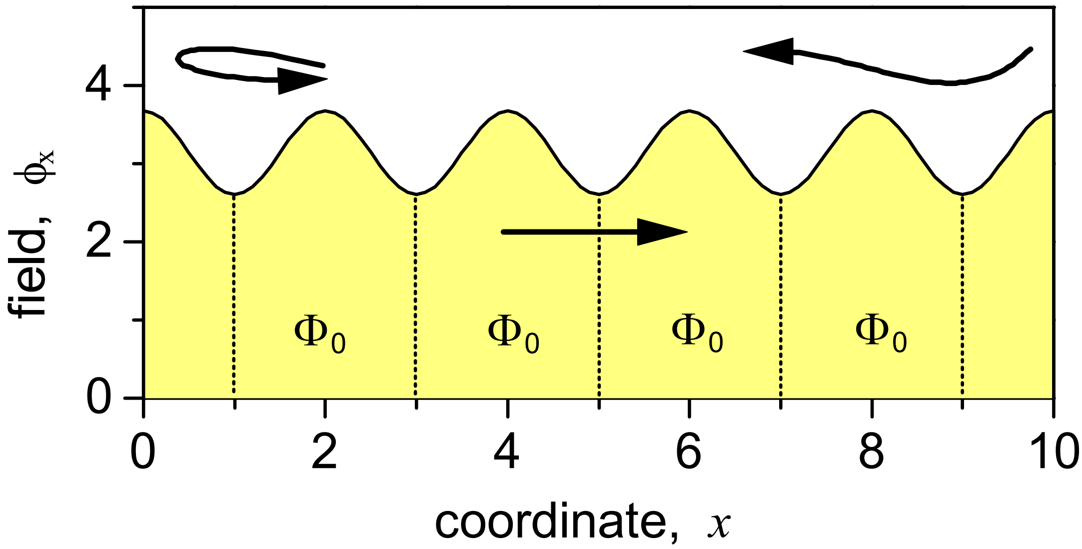
Flux-flow IVC

$$V_{\mathrm{FF}} = \frac{\Delta\Phi}{\Delta t} = \frac{H\Lambda L}{L/u} = H\Lambda u$$



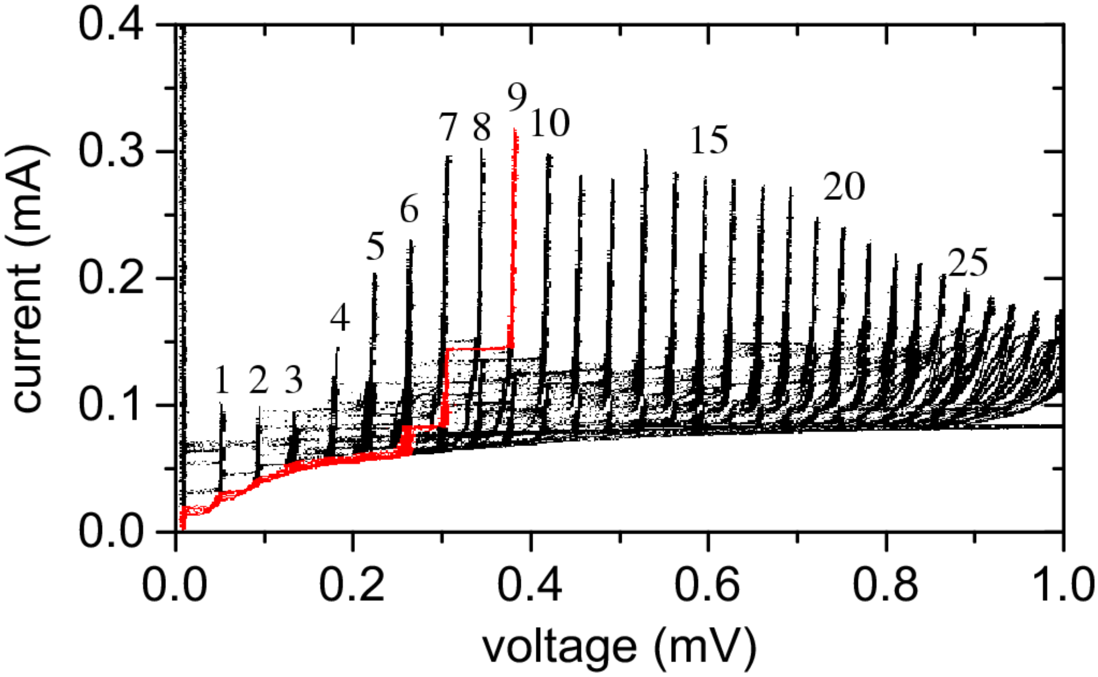
The maximum on the IVC at $u = \bar{c}_0$ is called a flux-flow resonance or Eck peak.
Application: tunable oscillators for the frequencies 50–$800\,\mathrm{GHz}$.

Interaction with edges



- The boundary conditions $\phi_x = h$ are not satisfied if we take running solutions $\phi(x - ut)$.
- $\Rightarrow$ we have to add "reflected wave" which propagates towards the middle of LJJ.
- This wave decays on the distances $\sim 1/\alpha$.
- $\alpha L \not\gg 1$ results in the formation of the standing wave.
- moving fluxons synchronize with this standing wave, resulting in geometrical resonances on the IVC at: $V_n^{\rm FS} = \Phi_0 \frac{\bar{c}_0}{2L} n$

Fiske Steps



- Linear theory ($H \gg 2$,$L \gg 1$) is developed.
- Non-linear theory (any H, any L) in the present state gives only the amplitude of resonances in 1-harmonic approximation.
- General nonlinear theory is not developed yet.
- Experimental IVC contains some features (shift or sub-families, fine structure of FSs) which are not explained.